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# THE LORENTZIAN VIVIANI CURVES AT THE LORENTZ-MINKOWSKI SPACE $R_1^3$

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**Abstract**

Spherical curves constitute an important class with widespread application. In Euclidean 3 – space, the intersection of a sphere of radius $2a$ centered at origin and a circular cylinder of radius $a$ centered at $(a,0,0)$ is known as the Viviani's (1622-1703) curve.

In this study, we define four Viviani curves which are intersections of the *Lorentzian sphere* of the Lorentz – Minkowski space $R_1^3$ with *circular cylinder with index* 1. We find the Frenet formulae, the curvature functions and the Frenet instantaneous rotation vectors for the Frenet frames defined on these curves.

**Keywords:** Lorentzian Sphere, Circular Cylinder with index 1, Lorentzian Cylinder, Hyperbolic Cylinder, Lorentzian Viviani Curves.

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## 1 Introduction

The most basic concepts in the construction of geometries are curves and surfaces. Considering that surfaces are defined with the help of curves, the importance of curves becomes apparent. The geometry of a plane or space curve is examined with the help of the Frenet framework defined at each point of the curve.

Some special curves defined in space or on a surface; robotics, kinematics, dynamics, computer-aided geometric design, simulation, textile, health, transportation, architecture, etc. Technological developments have increased with the use in different disciplines such as and these curves have played important roles in making our lives easier. For the usage areas of curves, see [1], [2] and [3].

Italian mathematician Vincenco Viviani proposed the problem of opening four equal windows on a hemispherical dome in 1692. The solution to this problem is given by the intersection of a sphere of radius and a right circular cylinder of radius. This intersection curve on the sphere and cylinder intersects itself at the end points of one diameter of the sphere. This curve, which is located on the sphere surface and resembles the figure-eight curve, is called the Viviani curve [4, 5, 22, 23].

The Viviani curve defined in Euclidean space is a special case of the Clelia curve (spherical spiral) [6]. General clelia curves including the generalized Viviani curve were defined in [5]. Different curves are also defined on the sphere of the Lorentz – Minkowski space $R_1^3$ [7, 8]. Barros and others have studied general helices in 3-dimensional Lorentzian spaceforms. In [9], Ferrandez et al. stated and proved the Lancret - type theorem for null generalized helices in 3 - dimensional Lorentz - Minkowski space $R_1^3$ . Furthermore, all helices at the Minkowski 3 – space $R_1^3$ were classified by Walrave [10]. Uğurlu et al. [11, 12] obtained the differential equations of timelike helices on the surface of the Lorentzian sphere with radius r and gave the parameterizations of these curves. They defined the Frenet and Darboux frames of these curves, derivative formulae, curvature functions, geodesic curvature function and instantaneous rotation vectors. In [21], they also defined

the differential equations of spirals defined on spheres of Minkowski space and expressed generalized parametrization of these special spherical curves. In [13], the differential equations of spirals on oblate and prolate spheroids of space were defined and their parameterizations were given. In [14], all Viviani curves at the Lorentz – Minkowski 3 – space $R_1^3$ defined and the geometries of the these curves examined with the help of Frenet frames.

In this study, four Viviani curves were defined on the Lorentzian sphere of radius $a$ of the 3 - dimensional Lorentz - Minkowski space $R_1^3$. Frenet derivative formulas, curvature and torsion functions were found, and Frenet instantaneous rotation vectors were obtained.

For more information about spherical curves and helical curves, see references [28, 29, 30].

In this study, vectors are shown in bold.

## 2 Fundamental Concepts

In this section, basic definitions and theorems at the Lorentz-Minkowski 3-space $R_1^3$ will be given.

The *Lorentz – Minkowski space* $R_1^3$ is the space $R^3$ endowed with the metric induced by the Lorentzian scalar product

$$\langle , \rangle : R^3 \times R^3 \to R$$

$$(\boldsymbol{x}, \boldsymbol{y}) \to \langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3 \tag{2.1}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ [15].

Let's point out here that this scalar product, like the Euclidean inner product, is bilineer, symmetric, nondegenerate. But it is indefinite, that is, it is not positive definite.

We say that a nonzero vector $x \in R_1^3$ is *spacelike if* $\langle x, x\rangle > 0$ or $x = 0$, *timelike* if $\langle x, x\rangle < 0$, *lightlike(null) if* $\langle x, x\rangle = 0$, respectively. The signature of a vector is defined by sign( $x$ ) = 1, 0 or -1 if $x$ is spacelike, lightlike or timelike, respectively. The *norm* of a vector $x \in R_1^3$ is defined by $\|x\| = \sqrt{|\langle x, x\rangle|}$.

$x$ is a *unit* spacelike (timelike) vector if $\langle x, x\rangle = 1$ $(\langle x, x\rangle = -1)$ [16].

Vectors $x$ and $y$ are called Lorentzian orthogonal if $\langle x, y\rangle = 0$. Two timelike vectors cannot be orthogonal. A vector perpendicular to a timelike vector is spacelike. Timelike vectors cannot be orthogonal to null vectors. A vector perpendicular to a spacelike vector can be spacelike or timelike or null. Further, two vectors that are symmetric in the Euclidean sense with respect to the light cone are perpendicular in the Lorentzian sense.

The *cross product* of any two $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ vectors of the space $R_1^3$ is defined by

$$x \times y = \det \begin{pmatrix} -e_1 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}, \tag{2.2}$$

where $\{e_1, e_2, e_3\}$ is the canonic basis of $R_1^3$. For any vector $u$, the equality

$$\langle u, x \times y\rangle = \det(u, x, y),$$

so that $x \times y$ is Lorentzian orthogonal to both $x$ and $y$. Moreover, if $x$ is a unit timelike vector, $y$ is a unit spacelike vector and $\langle x, y\rangle = 0$, $x \times y = z$, then by a direct calculation, we have

$$z \times x = -y, \; y \times z = -x, \; x \times y = z$$

[15, 17, 18, 19].

**Lemma 2. 1** Let $x=(x_1,x_2,x_3)$ , $y=(y_1,y_2,y_3)$ and $z=(z_1,z_2,z_3)$ be three vectors at the space $R_1^3$. In this case, the following equations are valid:

$$\text{i)} \quad x\times y=-y\times x$$

$$\text{ii)} \quad \langle x\times y,z\rangle=-\det(x,y,z)$$

$$\text{iii)} \quad (x\times y)\times z=-\langle x,z\rangle y+y\langle z,x\rangle$$

$$\text{iv)} \quad \langle x\times y,\ x\rangle=0 \ \text{ ve } \ \langle x\times y,\ y\rangle=0$$

$$\text{v)} \quad \langle x\times y,x\times y\rangle=-\langle x,x\rangle\langle y,y\rangle+\left(\langle x,y\rangle\right)^2 \ \text{(Lagrangian identity)}$$

[15, 17, 18].

**Definition 2. 2** Let $\alpha$: $I\to R_1^3$ be a regular curve. We say that any regular curve $\alpha$ is *spacelike* if for all $t\in I$, $\langle\alpha'(t),\alpha'(t)\rangle>0$; *timelike if* $\langle\alpha'(t),\alpha'(t)\rangle<0$, *lightlike (null) if* $\langle\alpha'(t),\alpha'(t)\rangle=0$ [17, 19].

**Definition 2. 3** Let $M$ be a regular surface in the space $R_1^3$ and let $P\in M$. We say that $M$ is *timelike surface* (resp., *spacelike surface*, *lightlike surface*) if for tangent plane $T_PM$ of the surface at point $M$, the induced metric

$$\langle\ ,\ \rangle_P : T_PM\times T_PM\to R$$

is indefined (resp., positive definite, degenerate).

The set of all spacelike vectors of unit length centered at the origin in space $R_1^3$ is called the *Lorentz sphere* (*de Sitter 2 – space*) of radius $r$ is denoted by

$$S_1^2(r)=\left\{x=(x_1,x_2,x_3)\in R_1^3:\ \langle x,\ x\rangle=r^2\right\}$$

If the Lorentzian scalar product is restricted on the Lorentzian sphere $S_1^2(r)$, the induced metric is indefined. This means that $S_1^2(r)$ is a timelike surface. Therefore, curves on the Lorentz sphere can have three characters.

The set of all timelike vectors of unit length centered at the origin in space $R_1^3$ is called the *unit hyperbolic 2 - sphere* of radius $r$ is denoted by

$$H_0^2(r) = \left\{ \boldsymbol{x} = (x_1, x_2, x_3) \in R_1^3 : \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -r^2 \right\}$$

The sphere $H_0^2(r)$ is the hyperboloid of two sheets of the Euclidean space $E^3$. The positive (hyperbolic 2 - space) and negative sheets of this sphere are denoted by

$$H^+(r) = \left\{ \boldsymbol{x} = (x_1, x_2, x_3) \in H_0^2(r) : \langle \boldsymbol{x}, \boldsymbol{e}_3 \rangle < 0 \right\}$$

and

$$H^-(r) = \left\{ \boldsymbol{x} = (x_1, x_2, x_3) \in H_0^2(r) : \langle \boldsymbol{x}, \boldsymbol{e}_3 \rangle > 0 \right\},$$

respectively [10, 15, 17, 18, 19].

If the Lorentzian scalar product is restricted to the hyperbolic sphere $H^+(r)$, the induced metric is positive definite. This means that $H^+(r)$ is a spacelike surface.

The set of all null vectors in space $R_1^3$ is called the *light cone* and denoted by

$$C^2 = \left\{ \boldsymbol{x} = (x_1, x_2, x_3) \in R_1^3 : \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0, \ \boldsymbol{x} \neq (0, 0, 0) \right\}.$$

The positive and negative sheets of the light cone $C^2$ are denoted by

$$C^+ = \left\{ \boldsymbol{x} = (x_1, x_2, x_3) \in C^2 : \langle \boldsymbol{x}, \boldsymbol{e}_3 \rangle < 0 \right\}$$

and

$$C^- = \{x = (x_1, x_2, x_3) \in C^2 : \langle x, e_3 \rangle > 0\},$$

respectively [10, 15, 17, 18, 19].

**Definition 2. 4 i)** Circular cylinder of index 1, $R_1^1 \times S^1$. The Cartesian equation of the of this cylinder is defined by $x_1^2 + x_2^2 = r^2$, $r > 0$. A parametrization for the cylinder is $X(u,v) = \left(\pm\sqrt{r^2 - v^2}, v, u\right)$; $o \le u \le 2\pi$, $v \in R$. This is a timelike surface [17,19,20].

**ii)** Lorentzian circular cylinder $S_1^1 \times R$. The Cartesian equation of the surface is $x_1^2 - x_3^2 = r^2$, $r > 0$. A parametrization of this cylinder is given by

$$X(u,v) = \left(\pm\sqrt{r^2 + u^2}, v, u\right); \; u, v \in R.$$

This is a timelike surface [17,19,20].

**iii)** Hyperbolic cylinder $H^1 \times R$. The Cartesian equation of the cylinder is given by $x_1^2 - x_3^2 = -r^2$, $r > 0$. A parametrization for this cylinder is

$$X(u,v) = \left(\pm\sqrt{u^2 - r^2}, v, u\right); \; u, v \in R.$$

This is a spacelike surface [17,19,20].

**Teorem 2. 5** Let $\alpha : I \subseteq R \to R_1^3$ be a regular timelike curve with arbitrary parameter $t$. In this case, the Frenet equations of the Frenet frame $\{T, N, B\}$ are given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & v(t)\kappa(t) & 0 \\ v(t)\kappa(t) & 0 & -v(t)\tau(t) \\ 0 & v(t)\tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}, \tag{2.3}$$

where $\langle T, T\rangle = -1$, $v(t) = \|\alpha'(t)\|$,

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}\ ,\ B(t) = -\frac{\alpha'(t)\times\alpha''(t)}{\|\alpha'(t)\times\alpha''(t)\|}\ ,\ N(t) = T(t)\times B(t) \tag{2.4}$$

and

$$\kappa(t) = \frac{\left\|\frac{d\alpha(t)}{dt}\times\frac{d^2\alpha(t)}{dt^2}\right\|}{\left\|\left(\frac{d\alpha(t)}{dt}\right)\right\|^3}\ ,\ \tau(t) = -\frac{\det\left(\frac{d\alpha(t)}{dt}, \frac{d^2\alpha(t)}{dt^2}, \frac{d^3\alpha(t)}{dt^3}\right)}{\left\|\frac{d\alpha(t)}{dt}\times\frac{d^2\alpha(t)}{dt^2}\right\|^2}. \tag{2.5}$$

By (2.3), the instantaneous rotation vector of the Frenet frame is defined by

$$\boldsymbol{f}(t) = v(t)\tau(t)\boldsymbol{T}(t) - v(t)\kappa(t)\boldsymbol{B}(t) \tag{2.6}$$

[15, 18,19].

**Teorem 2. 6** Let $\alpha : I \subseteq R \to R_1^3$ be a regular spacelike curve with arbitrary parameter $t$. In this case, the Frenet equations of the Frenet frame $\{T, N, B\}$ are given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & v(t)\kappa(t) & 0 \\ v(t)\kappa(t) & 0 & v(t)\tau(t) \\ 0 & v(t)\tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}, \tag{2.7}$$

where, $\langle N, N\rangle = -1$, $v(t) = \|\alpha'(t)\|$,

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}\ ,\ B(t) = -\frac{\alpha'(t)\times\alpha''(t)}{\|\alpha'(t)\times\alpha''(t)\|}\ ,\ N(t) = B(t)\times T(t) \tag{2.8}$$

and

$$\kappa(t)=\frac{\left\|\frac{d\alpha(t)}{dt}\times\frac{d^2\alpha(t)}{dt^2}\right\|}{\left\|\left(\frac{d\alpha(t)}{dt}\right)\right\|^3},\quad \tau(t)=-\frac{\det\left(\frac{d\alpha(t)}{dt},\frac{d^2\alpha(t)}{dt^2},\frac{d^3\alpha(t)}{dt^3}\right)}{\left\|\frac{d\alpha(t)}{dt}\times\frac{d^2\alpha(t)}{dt^2}\right\|^2}. \tag{2.9}$$

By (2.7), the instantaneous rotation vector of the Frenet frame is

$$\boldsymbol{f}(t)=-v(t)\tau(t)\boldsymbol{T}(t)+v(t)\kappa(t)\boldsymbol{B}(t) \tag{2.10}$$

[15,18,19].

**Theorem 2. 7** Let $\alpha: I\subseteq R \to R_1^3$ be a regular spacelike curve with arbitrary parameter t. In this case, the Frenet equations of the Frenet trihedron $\{T, N, B\}$ are given by

$$\begin{bmatrix}\boldsymbol{T}'(t)\\ \boldsymbol{N}'(t)\\ \boldsymbol{B}'(t)\end{bmatrix}=\begin{bmatrix}0 & v(t)\kappa(t) & 0\\ -v(t)\kappa(t) & 0 & v(t)\tau(t)\\ 0 & v(t)\tau(t) & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}(t)\\ \boldsymbol{N}(t)\\ \boldsymbol{B}(t)\end{bmatrix}, \tag{2.11}$$

where $\langle \boldsymbol{B},\boldsymbol{B}\rangle=-1$, $v(t)=\|\boldsymbol{\alpha}'(t)\|$,

$$\boldsymbol{T}(t)=\frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|},\ \boldsymbol{B}(t)=\frac{\boldsymbol{\alpha}'(t)\times\boldsymbol{\alpha}''(t)}{\|\boldsymbol{\alpha}'(t)\times\boldsymbol{\alpha}''(t)\|},\ \boldsymbol{N}(t)=\boldsymbol{T}(t)\times\boldsymbol{B}(t) \tag{2.12}$$

and the curvature functions are given by (2.9).

By (2.11), the instantaneous rotation vector of the Frenet trihedron is

$$\boldsymbol{f}(t)=v(t)\tau(t)\boldsymbol{T}(t)-v(t)\kappa(t)\boldsymbol{B}(t) \tag{2.13}$$

[15,18,19].

## 3 Lorentzian Viviani Curve I

In this chapter, we define the parametric equations and examine the geometry of the curve which is the intersection of the Lorentzian sphere of radius $a$ centered at the origin with the right circular cylinder of radius $a$ centered at $(2a, 0,0)$ in the Lorentz-Minkowski 3 - space.

A parameterization for this curve is given by

$$\alpha_1(t) = \left( a(2+\cos t),\ a\sin t,\ \pm 2\sqrt{2}\, a\cos\frac{t}{2} \right),\ t \in [-2\pi, 2\pi], \tag{3.1}$$

where $a$ is positive.The curve defined by (3.1) is called *Lorentzian Viviani curve I*.

For this curve, we have

$$v_1(t) = \| \alpha_1'(t) \| = \sqrt{\left| \langle \alpha_1'(t), \alpha_1'(t) \rangle \right|} = a\sqrt{|\cos t|} \tag{3.2}$$

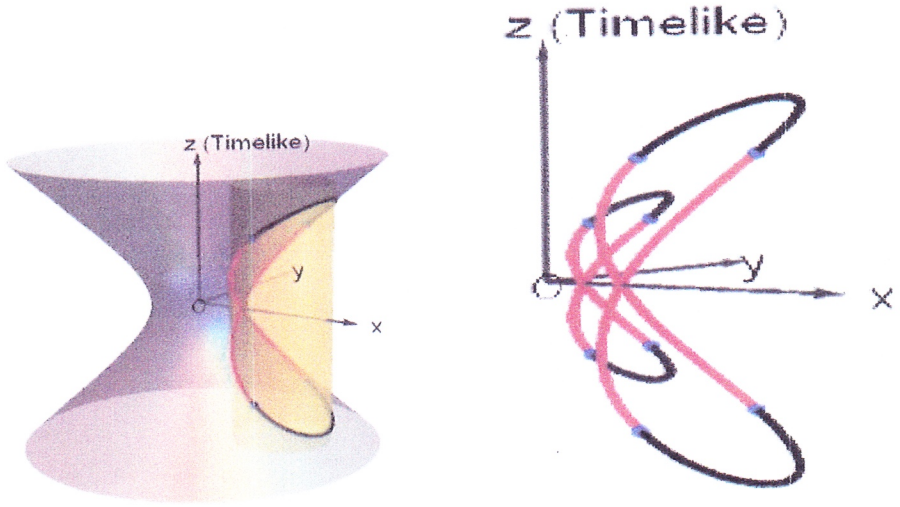
from

$$\alpha_1'(t) = \left( -a\sin t,\ a\cos t,\ \mp\sqrt{2}\, a\sin\frac{t}{2} \right) \tag{3.3}$$

and

$$\langle \alpha_1'(t), \alpha_1'(t) \rangle = a^2 \left( 1 - 2\left( \frac{1-\cos t}{2} \right) \right) = a^2 \cos t.$$

That is, $\alpha_1(t)$ is not parametrized by the arc length parameter. If $\cos t > 0$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, the curve arc is spacelike, if $\cos t < 0$, $\frac{\pi}{2} < t < \frac{3\pi}{2}$, the curve arc is timelike, and if $\cos t = 0$, $t = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$, the tangents of the curve are lightlike vectors.




(a) (b)

Fig. 3. 1: Lorentzian Viviani Curve I, spacelike (black) and timelike (red) arcs, and lightlike points (blue), (a) $a=1$, (b) $a=1$ and $a=2$.

**Theorem 3. 1** Let $\alpha_1 : I \subseteq \mathbb{R} \to S_1^2(a)$ be regular Lorentzian Viviani curve I with arbitrary parameter $t$. In this case, the Frenet equations of the Frenet frame $\{T, N, B\}$ are given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\sqrt{|1-3\cos t|}}{2a\cos t} & 0 \\ \varepsilon\dfrac{\sqrt{|1-3\cos t|}}{2a\cos t} & 0 & \mp\dfrac{3\sqrt{2}\sin\frac{t}{2}\sqrt{\cos t}}{a|1-3\cos t|} \\ 0 & \mp\dfrac{3\sqrt{2}\sin\frac{t}{2}\sqrt{\cos t}}{a|1-3\cos t|} & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} \quad (3.4)$$

for $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{1-3\cos t}}{2a\cos t} & 0 \\ \frac{\sqrt{1-3\cos t}}{2a\cos t} & 0 & \pm\frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{-\cos t}}{a(1-3\cos t)} \\ 0 & \mp\frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{-\cos t}}{a(1-3\cos t)} & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} \quad (3.5)$$

for $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

**Proof.** By (3.2) and (3.3) the unit tangent vector field is

$$T(t) = \frac{1}{\sqrt{|\cos t|}}\left(-\sin t,\ \cos t,\ \mp\sqrt{2}\sin\frac{t}{2}\right).$$

*i*) The curve $\alpha_1(t)$ is spacelike in the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Thus, we have

$$T(t) = \frac{1}{\sqrt{\cos t}}\left(-\sin t,\ \cos t,\ \mp\sqrt{2}\sin\frac{t}{2}\right), \quad (3.6)$$

$$T'(t) = \left(-\frac{\cos^2 t+1}{2(\cos t)^{3/2}},\ -\frac{\sin t}{2\sqrt{\cos t}},\ \mp\frac{\sqrt{2}}{2}\frac{\cos\frac{t}{2}}{(\cos t)^{3/2}}\right) \quad (3.7)$$

and

$$\langle \boldsymbol{T}'(t), \boldsymbol{T}'(t) \rangle = \frac{1}{4}\left(\cos^2 t + 1\right)^2 \sec^3(t) + \frac{1}{4}\sin^2 t \sec t - \frac{1}{2}\cos^2 \frac{t}{2} \sec^3 t$$

$$= \frac{1}{4}\sec t\left(3 - \sec t\right).$$

The solution of the equation

$$\langle \boldsymbol{T}'(t), \boldsymbol{T}'(t) \rangle = 0 \Rightarrow \frac{1}{4}\sec t\left(3 - \sec t\right) = 0$$

is $t = \pm \arccos\frac{1}{3}$. In these points, the vectors $\boldsymbol{T}'(t)$ are lightlike (Figure 3. 2).

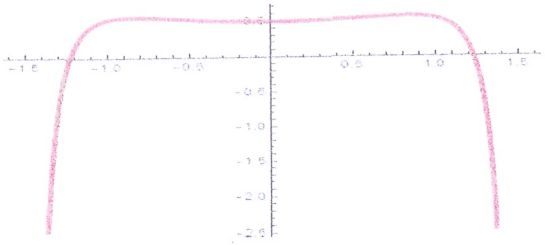


Fig. 3. 2: The function $\langle \boldsymbol{T}'(t), \boldsymbol{T}'(t) \rangle$ in $-\frac{\pi}{2} < t < \frac{\pi}{2}$

By a direct calculation, we have

$$\boldsymbol{\alpha}_1''(t) = \left(-a\cos t, -a\sin t, \mp\frac{\sqrt{2}}{2}a\cos\frac{t}{2}\right),$$

$$\boldsymbol{\alpha}_1'''(t) = \left(a\sin t, -a\cos t, \pm\frac{\sqrt{2}}{4}a\sin\frac{t}{2}\right).$$

If we find the cross product of the first and second derivatives and take the length of the vector, we obtain

$$\| \alpha_1'(t) \times \alpha_1''(t) \| = \sqrt{\left| \langle \alpha_1'(t) \times \alpha_1''(t),\ \alpha_1'(t) \times \alpha_1''(t) \rangle \right|} = \frac{a^2}{\sqrt{2}} \sqrt{\left| 3\sin^2 \frac{t}{2} - 1 \right|}$$

from

$$\alpha_1'(t) \times \alpha_1''(t) = \begin{vmatrix} -e_1 & -e_2 & e_3 \\ -a\sin t & a\cos t & \mp\sqrt{2}\, a \sin\frac{t}{2} \\ -a\cos t & -a\sin t & \mp\frac{\sqrt{2}}{2}\, a\cos\frac{t}{2} \end{vmatrix},$$

$$= \left( \pm\frac{\sqrt{2}}{2} a^2 \cos t \cos\frac{t}{2} \pm \sqrt{2}\, a^2 \sin t \sin\frac{t}{2},\ \pm\frac{\sqrt{2}}{2} a^2 \sin t \cos\frac{t}{2} \mp \sqrt{2}\, a^2 \cos t \sin\frac{t}{2},\ a^2 \right),$$

$$\langle \alpha_1'(t) \times \alpha_1''(t),\ \alpha_1'(t) \times \alpha_1''(t) \rangle = \left( \pm\frac{\sqrt{2}}{2} a^2 \cos t \cos\frac{t}{2} \pm \sqrt{2} a^2 \sin t \sin\frac{t}{2} \right)^2 +$$

$$\left( \pm\frac{\sqrt{2}}{2} a^2 \sin t \cos\frac{t}{2} \mp \sqrt{2} a^2 \cos t \sin\frac{t}{2} \right)^2 - a^4$$

$$= \frac{a^4}{2} \left( 3\sin^2 \frac{t}{2} - 1 \right).$$

Then, it is seen that $t = \pm \arccos\frac{1}{3}$ from

$$\langle \alpha_1'(t) \times \alpha_1''(t),\ \alpha_1'(t) \times \alpha_1''(t) \rangle = 0.$$

(Figure 3. 3)

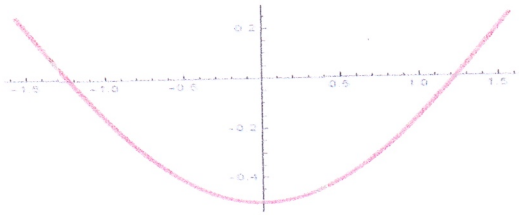


Fig. 3. 3: The function $\langle \alpha_1'(t) \times \alpha_1''(t), \alpha_1'(t) \times \alpha_1''(t) \rangle$, $a = 1$.

Therefore, the unit binormal vector field is

$$B(t) = \frac{1}{\sqrt{|1-3\cos t|}} \left( \pm \frac{\cos\frac{3t}{2} - 3\cos\frac{t}{2}}{\sqrt{2}}, \mp 2\sqrt{2}\sin^3\frac{t}{2}, -2 \right). \tag{3.8}$$

Lets denote compenents of unit priciple normal vector field

$$N(t) = \frac{\sqrt{2}}{\sqrt{\cos t}\sqrt{\left|3\sin^2\frac{t}{2}-1\right|}} \begin{vmatrix} -e_1 & -e_2 & e_3 \\ \mp\frac{\sqrt{2}}{2}\cos t\cos\frac{t}{2} \mp \sqrt{2}\sin t\sin\frac{t}{2} & \mp\frac{\sqrt{2}}{2}\sin t\cos\frac{t}{2} \pm \sqrt{2}\cos t\sin\frac{t}{2} & -1 \\ -\sin t & \cos t & \mp\sqrt{2}\sin\frac{t}{2} \end{vmatrix}$$

by $N_i$, $i = 1,2,3$. Thus, we find

$$N_1 = \frac{\sqrt{2}}{\sqrt{\cos t}\sqrt{\left|3\sin^2\frac{t}{2}-1\right|}} \left( -\sin t\sin\frac{t}{2}\cos\frac{t}{2} + 2\cos t\sin\frac{t}{2}\sin\frac{t}{2} - \cos t \right)$$

$$= -\frac{1+\cos^2 t}{\sqrt{2}\sqrt{\cos t}\sqrt{\left|3\sin^2\frac{t}{2}-1\right|}},$$

$$N_2 = \frac{\sqrt{2}}{\sqrt{\cos t}\sqrt{\left|3\sin^2\frac{t}{2}-1\right|}}\left(\cos t\cos\frac{t}{2}\sin\frac{t}{2}+2\sin t\sin\frac{t}{2}\sin\frac{t}{2}-\sin t\right)$$

$$= -\frac{\sin t\cos t}{\sqrt{2}\sqrt{\cos t}\sqrt{\left|3\sin^2\frac{t}{2}-1\right|}}$$

and

$$N_3 = \frac{\sqrt{2}}{\sqrt{\cos t}\sqrt{\left|3\sin^2\frac{t}{2}-1\right|}}\left(\mp\frac{\sqrt{2}}{2}\cos t\cos t\cos\frac{t}{2}\mp\sqrt{2}\sin t\cos t\sin\frac{t}{2}\mp\frac{\sqrt{2}}{2}\sin t\sin t\cos\frac{t}{2}\pm\sqrt{2}\sin t\cos t\sin\frac{t}{2}\right)$$

$$= \mp\frac{\cos\frac{t}{2}}{\sqrt{\cos t}\sqrt{\left|3\sin^2\frac{t}{2}-1\right|}}.$$

Consequently, we have

$$N(t) = \frac{1}{\sqrt{\cos t}\sqrt{\left|3\sin^2\frac{t}{2}-1\right|}}\left(-\frac{1+\cos^2 t}{\sqrt{2}}, -\frac{\sin t\cos t}{\sqrt{2}}, \mp\cos\frac{t}{2}\right)$$

or

$$N(t) = \frac{\sqrt{\cos t}}{\sqrt{|1-3\cos t|}}\left(-\frac{(\cos 2t+3)\sec t}{2}, -\sin t, \mp\sqrt{2}\cos\frac{t}{2}\sec t\right). \qquad (3.9)$$

By (2.9), the curvature and torsion functions are found in the forms

$$\kappa(t)=\frac{\frac{a^2}{\sqrt{2}}\sqrt{\left|3\sin^2\frac{t}{2}-1\right|}}{a^3\left(\cos t\right)^{3/2}}$$

$$\kappa(t)=\frac{\sqrt{\left|1-3\cos t\right|}}{2a\left(\cos t\right)^{3/2}} \tag{3.10}$$

and

$$\tau(t)=\mp\frac{3\sqrt{2}}{4}a^3\sin\frac{t}{2}\frac{2}{a^4\left|3\sin^2\frac{t}{2}-1\right|}$$

$$\tau(t)=\mp\frac{3\sqrt{2}\sin\frac{t}{2}}{a\left|1-3\cos t\right|} \tag{3.11}$$

from

$$\left\langle \alpha_1'(t)\times\alpha_1''(t),\alpha_1'''\right\rangle=\pm\frac{\sqrt{2}}{2}a^3\sin t\cos t\cos\frac{t}{2}\pm\sqrt{2}a^3\sin^2 t\sin\frac{t}{2}\mp$$
$$\frac{\sqrt{2}}{2}a^3\sin t\cos t\cos\frac{t}{2}\pm\sqrt{2}a^3\cos^2 t\sin\frac{t}{2}\mp\frac{\sqrt{2}}{4}a^3\sin\frac{t}{2}$$

$$=\pm\sqrt{2}\,a^3\sin\frac{t}{2}\mp\frac{\sqrt{2}}{4}a^3\sin\frac{t}{2}$$

$$=\pm\frac{3\sqrt{2}}{4}a^3\sin\frac{t}{2},$$

respectively.

For $t = 0$ the torsion is zero. Since the torsion function measures the amount of deviation of the curve from the tangent plane, the sign here indicates that the curve is on the opposite side of the tangent plane (Figure 3. 4).

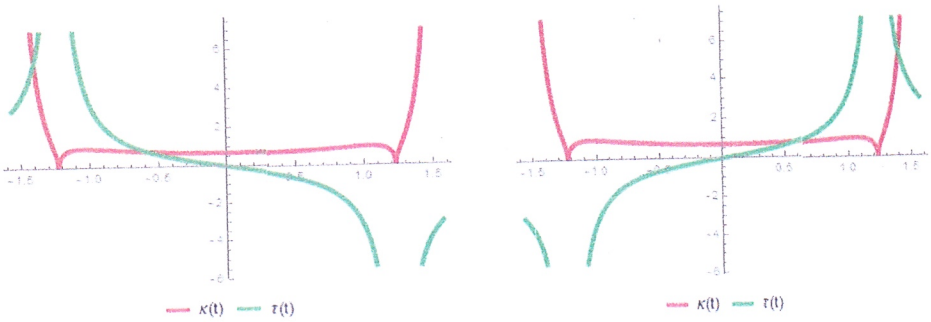



Fig. 3. 4: Curvature and torsion functions of the curve $\alpha_1(t)$ at the open interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $a = 1$.

From figure 3. 2 and figure 3. 3, the vector $T'(t)$ is spacelike in interval $\left(-\arccos\frac{1}{3}, \arccos\frac{1}{3}\right)$. In this domain, the characters of the principle normal and binormal change while the curvature and torsion functions do not change. Now, let's take the spacelike arc of the curve. Then, the Frenet derivative formulas of the frame $\{T(t), N(t), B(t)\}$ are given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{|1-3\cos t|}}{2\cos t} & 0 \\ \frac{\varepsilon\sqrt{|1-3\cos t|}}{2\cos t} & 0 & \mp\frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{\cos t}}{|1-3\cos t|} \\ 0 & \mp\frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{\cos t}}{|1-3\cos t|} & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix} \quad (3.12)$$

from equations (3.10) and (3.11), with $\varepsilon = 1$ at the interval where the vector $T'(t)$ is timelike and $\varepsilon = -1$ at the interval where it is spacelike.

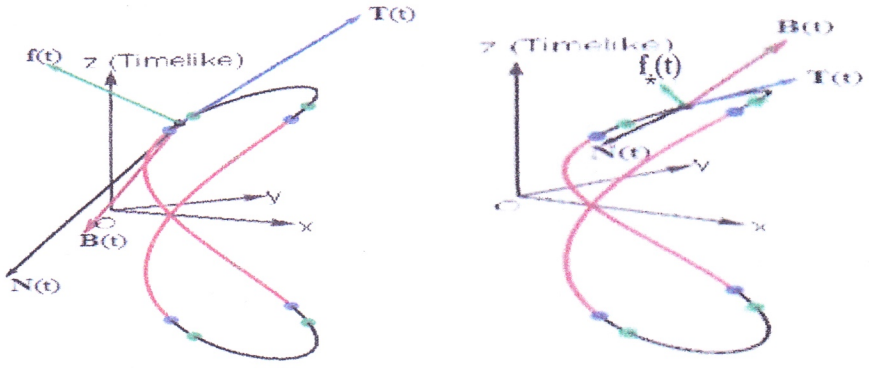
**Corollary 3. 2** For $t \in \left(-\frac{\pi}{2}, -\arccos\frac{1}{3}\right) \cup \left(\arccos\frac{1}{3}, \frac{\pi}{2}\right)$, the vector $T'(t)$ is timelike. Hence, the instantaneous rotation vector of the Frenet frame of the spacelike arc of the curve $\alpha_1(t)$ is given by

$$f(t) = \pm 3\sqrt{2}\ \frac{\sqrt{\cos t}\sin\frac{t}{2}}{1-3\cos t}\,T(t) + \frac{\sqrt{1-3\cos t}}{2\cos t}\,B(t). \tag{3.13}$$

For $t \in \left(-\arccos\frac{1}{3}, \arccos\frac{1}{3}\right)$, the vector $T'(t)$ is spacelike. Then the instantaneous rotation vectors of the Frenet frame of the spacelike arc of the curve $\alpha_1(t)$ is

$$f_*(t) = \mp 3\sqrt{2}\ \frac{\sqrt{\cos t}\sin\frac{t}{2}}{3\cos t - 1}\,T(t) - \frac{\sqrt{3\cos t - 1}}{2\cos t}\,B(t). \tag{3.14}$$

When the + sign is taken in (3.1), the Frenet trihedron and instantaneous rotation vectors are seen in figure 3. 5.




(a) (b)

Fig. 3. 5: (a) The Frenet trihedron and instantaneous rotation vectors of the spacelike arc of the curve $\alpha_1(t)$, $T'(t)$ is timelike, $t \in \left(-\frac{\pi}{2}, -\cos^{-1}\frac{1}{3}\right)$,

(b) The Frenet trihedron and instantaneous rotation vectors of the spacelike arc of the curve $\alpha_1(t)$, $T'(t)$ is spacelike $t \in \left(-\cos^{-1}\frac{1}{3}, \cos^{-1}\frac{1}{3}\right)$.

Since the curve $\alpha_1(t)$ is not of single character, both non-null frames and null frames are defined on it. Null frames are defined for $t = \cos^{-1}\frac{1}{3}$, green colored points and $t = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$, blue colored points as seen in figure 3.5. The frame at these points are similar to the null frames in [24, 25]. However at the blue colored points on the curve $\alpha_1(t)$, the character of the curve changes and at the green colored points, the characteristics of the vectors N and B exchange.

*ii*) Let $\frac{\pi}{2} < t < \frac{3\pi}{2}$. In this case, we have

$$T(t)=\frac{1}{\sqrt{-\cos t}}\left(-\sin t,\ \cos t,\ \mp\sqrt{2}\sin\frac{t}{2}\right) \tag{3.15}$$

and

$$\langle T(t),\ T(t)\rangle=\frac{1-2\sin^2\frac{t}{2}}{-\cos t}=\frac{\cos t}{-\cos t}=-1<0\,.$$

This means that the curve $\alpha_1(t)$ is timelike.

For the vector $T(t)$ we find

$$T'(t)=\left(-\frac{\cos t}{\sqrt{-\cos t}}+\frac{\sin^2 t}{2(-\cos t)^{3/2}},\ -\frac{\sin t}{\sqrt{-\cos t}}-\frac{\cos t\sin t}{2(-\cos t)^{3/2}},\ -\frac{\cos\frac{t}{2}}{\sqrt{2}\sqrt{-\cos t}}+\frac{\sin\frac{t}{2}\sin t}{\sqrt{2}(-\cos t)^{3/2}}\right)$$

$$T'(t)=\frac{1}{\sqrt{2}\cos t\sqrt{-\cos t}}\left(\frac{1+\cos^2 t}{\sqrt{2}},\ \frac{\sin t\cos t}{\sqrt{2}},\ \pm\cos\frac{t}{2}\right)$$

$$T'(t)=\left(\frac{1}{2}\sqrt{-\cos t}\left(2+\tan^2 t\right),\ \frac{\sin t}{2\sqrt{-\cos t}},\ \mp\frac{\cos\frac{t}{2}+\sin\frac{t}{2}\tan t}{\sqrt{2}\sqrt{-\cos t}}\right) \tag{3.16}$$

and

$$\langle T'(t),T'(t)\rangle=-\frac{1}{4}\sin t\tan t-\frac{1}{4}\cos t\left(2+\tan^2 t\right)^2+\frac{1}{2}\sec t\left(\cos\frac{t}{2}+\sin\frac{t}{2}\tan t\right)^2$$

$$=\frac{1}{4}(-3+\sec t)\sec t\,.$$

Thus, the vector $\boldsymbol{T'(t)}$ is spacelike in the interval $\frac{\pi}{2} < t < \frac{3\pi}{2}$ ( Figure 3. 6 ).

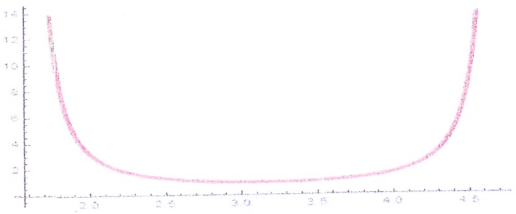


Fig. 3. 6: The function $\langle \boldsymbol{T'(t)}, \boldsymbol{T'(t)} \rangle$ in $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

Since $3\sin^2\frac{t}{2} - 1 = \frac{1-3\cos t}{2} > 0$ in interval $\frac{\pi}{2} < t < \frac{3\pi}{2}$, the vector field

$$\boldsymbol{B}(t) = \frac{1}{\sqrt{1-3\cos t}}\left( \pm \frac{\cos\frac{3t}{2} - 3\cos\frac{t}{2}}{\sqrt{2}}, \mp 2\sqrt{2}\sin^3\frac{t}{2}, -2 \right) \tag{3.17}$$

is spacelike and the principle normal vector field is given by

$$N(t) = \frac{\sqrt{-\cos t}}{\sqrt{1-3\cos t}}\left( \frac{(\cos 2t + 3)\sec t}{2}, \sin t, \pm\sqrt{2}\cos\frac{t}{2}\sec t \right). \tag{3.18}$$

By (2.9), the curvature and torsion functions of curve are obtained in the forms given by

$$\kappa(t) = \frac{\sqrt{1-3\cos t}}{2a(-\cos t)^{3/2}} \tag{3.19}$$

and

$$\tau(t) = \pm \frac{3\sqrt{2}\sin\frac{t}{2}}{a(1-3\cos t)}, \tag{3.20}$$

respectively (Figure 3. 7).

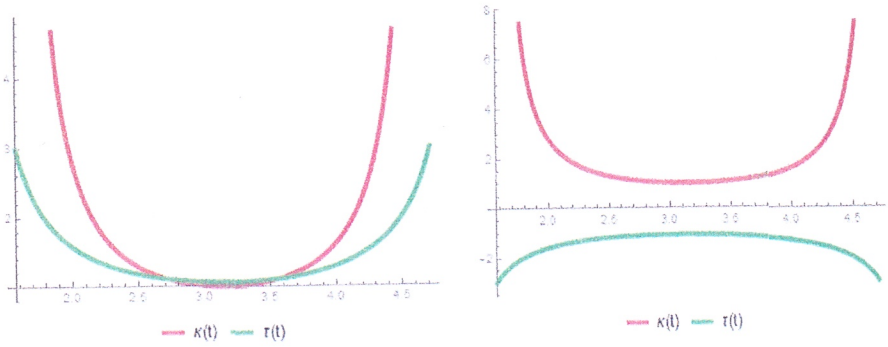



Fig. 3. 7: The curvature and torsion functions of the timelike arc of the curve $\alpha_1(t)$, $\frac{\pi}{2} < t < \frac{3\pi}{2}$, $a = 1$.

Thus, the Frenet formulas of the timelike arc of the curve $\alpha_1(t)$ is given by

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{1-3\cos t}}{2\cos t} & 0 \\ \frac{\sqrt{1-3\cos t}}{2\cos t} & 0 & \pm\frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{-\cos t}}{1-3\cos t} \\ 0 & \mp\frac{3\sqrt{2}\sin\frac{t}{2}\sqrt{-\cos t}}{1-3\cos t} & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}. \tag{3.21}$$

**Corollary 3. 3** The Frenet instantaneous rotation vector of the timelike arc of the curve $\alpha_1(t)$ in the interval $\frac{\pi}{2} < t < \frac{3\pi}{2}$ can be written in the form

$$f_1(t) = \pm \frac{3\sqrt{2}\sqrt{-\cos t}\sin\frac{t}{2}}{1-3\cos t} T(t) - \frac{\sqrt{-\cos t}\sqrt{1-3\cos t}}{2(-\cos t)^{3/2}} B(t) \qquad (3.22)$$

(Figure 3. 8).

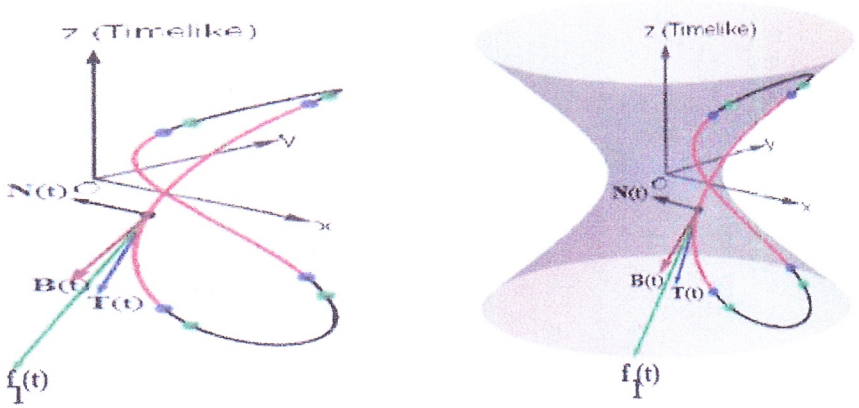



Fig. 3. 8: Frenet trihedron and the instantaneous rotation vector of the timelike arc of the curve $\alpha_1(t)$

**Corollary 3. 4** The Frenet formulas of the timelike arc of the curve $\alpha_1(t)$ can be written in the form

$$T'(t) = f_1(t) \times T(t), \quad N'(t) = f_1(t) \times N(t), \quad B'(t) = f_1(t) \times B(t)$$

by the vector $f_1(t)$.

**Proof.** It is sufficient to multiply the Frenet vectors by $f_1(t)$ vectorically in the sense of Lorentz.

## 4 Lorentzian Viviani Curve II

In this chapter, we define the parametric equations and examine the geometry of the curve which is the intersection of the Lorentzian sphere of the radius $a$ centered at the origin with Lorentzian cylinder of radius $a$ centered at $(2a, 0, 0)$ and radius $a$ at the Lorentz -Minkowski 3 - space.

A parametrization for this curve is given by

$$\alpha_2(t) = \left( a(2-\cosh t), \pm 2\sqrt{2}\, a \sinh\frac{t}{2}, a\sinh t \right), \quad t \in \mathbb{R} \tag{4.1}$$

where $a$ is positive. The curve defined by (4.1) is called *Lorentzian Viviani curve II.*

For the curve (4.1), we have

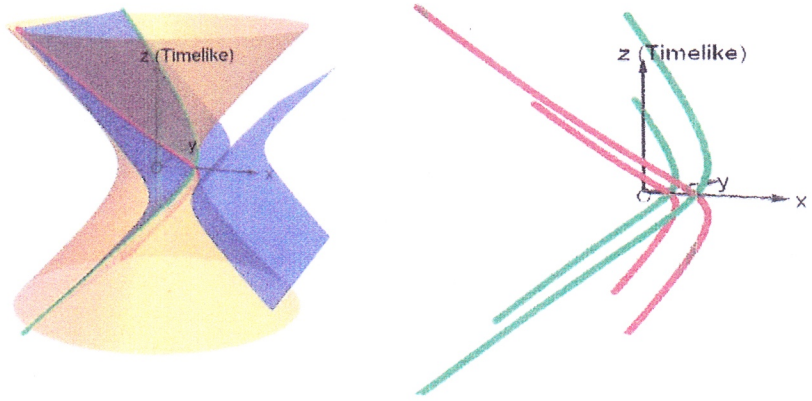
$$v_2(t) = \| \alpha_2'(t) \| = \sqrt{|\langle \alpha_2'(t), \alpha_2'(t) \rangle|} = a\sqrt{\cosh t} \tag{4.2}$$

from

$$\alpha_2'(t) = \left( -a\sinh t, \pm\sqrt{2}\, a\cosh\frac{t}{2}, a\cosh t \right). \tag{4.3}$$

By (4.2), $\alpha_2(t)$ is spacelike (Figure 4. 1) and it is not parametrized by the arc length.

If the second component of the curve $\alpha_2(t)$ is taken as +, the green colored curve arc is obtained and if it is taken as - , the red curve arc is obtained, as seen in figüre (4.1).




(a) (b)

Fig. 4. 1: Lorentzian Viviani curve II (a) $a=1$, (b) $a=1$ and $a = 2$.

**Theorem 4. 1** Let $\alpha_2 : I \subseteq \mathbb{R} \to S_1^2(a)$ be regular spacelike Lorentzian Viviani curve II with arbitrary parameter t. In this case, the Frenet formulae of the Frenet frame $\{T, N, B\}$ are given by

$$\left.\begin{aligned} T'(t) &= \frac{1}{2}\frac{\sqrt{3\cosh t+1}}{\cosh^2 t}N(t) \\ N'(t) &= -\frac{1}{2}\frac{\sqrt{3\cosh t+1}}{\cosh^2 t}T(t) \pm 3\sqrt{2}\frac{\cosh\frac{t}{2}\sqrt{\cosh t}}{3\cosh t+1}B(t) \\ B'(t) &= \pm 3\sqrt{2}\frac{\cosh\frac{t}{2}\sqrt{\cosh t}}{3\cosh t+1}N(t) \end{aligned}\right\}. \qquad (4.4)$$

**Proof.** By (4.2) and (4.3) we have

$$T(t) = \frac{1}{\sqrt{\cosh t}}\left(-\sinh t,\ \pm\sqrt{2}\cosh\frac{t}{2},\ \cosh t\right). \tag{4.5}$$

The derivatives of components of vector field $T(t)$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\sinh t}{\sqrt{\cosh t}}\right) = -\frac{\cosh t\sqrt{\cosh t} - \sinh^2 t}{2\cosh^{3/2} t} = \frac{\sinh^2 t - 2\cosh^2 t}{2\cosh^{3/2} t} = -\frac{\cosh^2 t + 1}{2\cosh^{3/2} t},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\pm\frac{\sqrt{2}\cosh\frac{t}{2}}{\sqrt{\cosh t}}\right) = \pm\frac{\sinh\frac{t}{2}\cosh t - \sinh t\cosh\frac{t}{2}}{\sqrt{2}\cosh^{3/2} t} = \mp\frac{\sinh\frac{t}{2}}{\sqrt{2}\cosh^{3/2} t}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\cosh t}{\sqrt{\cosh t}}\right) = \frac{2\sinh t\cosh t - \sinh t\cosh t}{2\cosh^{3/2} t} = \frac{\sinh t\cosh t}{2\cosh^{3/2} t} = \frac{\sinh t}{2\sqrt{\cosh t}},$$

respectively. Therefore, we have

$$T'(t) = \left(-\frac{\cosh^2 t + 1}{2(\cosh t)^{3/2}},\ \mp\frac{\sinh\frac{t}{2}}{\sqrt{2}(\cosh t)^{3/2}},\ \frac{\sinh t}{2\sqrt{\cosh t}}\right) \tag{4.6}$$

and

$$\langle T'(t), T'(t)\rangle = \left(\frac{\sinh^2 t - 2\cosh^2 t}{2\cosh^{3/2} t}\right)^2 + \frac{\sinh^2\frac{t}{2}}{2\cosh^3 t} - \frac{\sinh^2 t}{4\cosh t}$$

$$= \frac{\cosh t(3\cosh t + 1)}{4\cosh^3 t}.$$

Since $\cosh t > 1$, $\boldsymbol{T}'(t)$ is spacelike and it seen that

$$\|\boldsymbol{T}'(t)\| = \sqrt{\frac{\cosh t(3\cosh t+1)}{4\cosh^3 t}} = \frac{1}{2}\frac{\sqrt{3\cosh t+1}}{\cosh t}.$$

By a straightforward calculation, we have

$$\boldsymbol{\alpha}_2''(t) = \left(-a\cosh t, \pm\frac{a}{\sqrt{2}}\sinh\frac{t}{2}, a\sinh t\right)$$

and

$$\boldsymbol{\alpha}_2'(t)\times\boldsymbol{\alpha}_2''(t) = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 \\ -a\sinh t & \pm\sqrt{2}\,a\cosh\frac{t}{2} & a\cosh t \\ -a\cosh t & \pm\frac{a}{\sqrt{2}}\sinh\frac{t}{2} & a\sinh t \end{vmatrix}$$

$$= a^2\left(\mp\sqrt{2}\sinh t\cosh\frac{t}{2}\pm\frac{1}{\sqrt{2}}\cosh t\sinh\frac{t}{2},\ 1,\ \mp\frac{1}{\sqrt{2}}\sinh t\sinh\frac{t}{2}\pm\sqrt{2}\cosh t\cosh\frac{t}{2}\right)$$

$$= a^2\left(\mp\frac{\sqrt{2}}{4}\sinh\frac{3t}{2}\mp\frac{3\sqrt{2}}{4}\sinh\frac{t}{2},\ 1,\ \pm\frac{\sqrt{2}}{4}\cosh\frac{3t}{2}\pm\frac{3\sqrt{2}}{4}\cosh\frac{t}{2}\right)$$

$$\boldsymbol{\alpha}_2'(t)\times\boldsymbol{\alpha}_2''(t) = a^2\left(\mp\frac{\sqrt{2}}{4}\sinh\frac{3t}{2}\mp\frac{3\sqrt{2}}{4}\sinh\frac{t}{2},\ 1,\ \pm\sqrt{2}\cosh^3\frac{t}{2}\right).$$

Therefore, it is seen that

$$\|\boldsymbol{\alpha}_2'(t)\times\boldsymbol{\alpha}_2''(t)\| = \frac{a^2}{2}\sqrt{3\cosh t+1}.$$

Consequently, by (2.9) the curvature function and unit priciple normal vector field are found in the forms

$$\kappa(t)=\frac{\|T'(t)\|}{\|\alpha_2'(t)\|}=\frac{\frac{1}{2}\frac{\sqrt{3\cosh t+1}}{\cosh t}}{a\sqrt{\cosh t}}=\frac{1}{2}\frac{\sqrt{3\cosh t+1}}{a(\cosh t)^{3/2}}$$

$$\kappa(t)=\frac{1}{2}\frac{\sqrt{3\cosh t+1}}{a(\cosh t)^{3/2}}\ . \tag{4.7}$$

and

$$N(t)=\frac{2\cosh t}{\sqrt{3\cosh t+1}}\left(\frac{\sinh^2 t-2\cosh^2 t}{2(\cosh t)^{3/2}},\mp\frac{\sinh\frac{t}{2}}{\sqrt{2}(\cosh t)^{3/2}},\frac{\sinh t}{2\sqrt{\cosh t}}\right)$$

$$=\left(\frac{\sinh^2 t-2\cosh^2 t}{\sqrt{\cosh t}\sqrt{3\cosh t+1}},\mp\frac{\sqrt{2}\sinh\frac{t}{2}}{\sqrt{\cosh t}\sqrt{3\cosh t+1}},\frac{\sinh t\cosh t}{\sqrt{\cosh t}\sqrt{3\cosh t+1}}\right)$$

$$N(t)=\left(-\frac{\sqrt{\cosh t}(\cosh 2t+3)\operatorname{sech} t}{2\sqrt{3\cosh t+1}},\mp\frac{\sqrt{2}\sqrt{\cosh t}\sinh\frac{t}{2}\operatorname{sech} t}{\sqrt{3\cosh t+1}},\frac{\sqrt{\cosh t}\sinh t}{\sqrt{3\cosh t+1}}\right). \tag{4.8}$$

The cross product of the vector fields $T(t)$ and $N(t)$ give the unit binormal vector field

$$B(t)=\frac{1}{\sqrt{3\cosh t+1}}\left(\mp\frac{\sinh\frac{3t}{2}+3\sinh\frac{t}{2}}{\sqrt{2}},\ 2\ ,\pm 2\sqrt{2}\cosh^3\frac{t}{2}\right). \tag{4.9}$$

Since $\langle B(t), B(t)\rangle = -1$, $B(t)$ is timelike. If the third derivative of the curve $\alpha_2(t)$ is found and take the mixed product of the first three derivatives is taken, it is seen that

$$\det\left(\alpha_2'(t), \alpha_2''(t), \alpha_2'''(t)\right) = -\left\langle \alpha_2'(t) \times \alpha_2''(t), \alpha_2'''(t)\right\rangle$$

$$= -a^3\left( \pm \sqrt{2}\sinh^2 t \cosh\frac{t}{2} \mp \frac{1}{\sqrt{2}}\sinh t\cosh t\sinh\frac{t}{2} \pm \frac{1}{2\sqrt{2}}\cosh\frac{t}{2}\right.$$

$$\left. \pm \frac{1}{\sqrt{2}}\sinh t\cosh t\sinh\frac{t}{2} \mp \sqrt{2}\cosh^2 t\cosh\frac{t}{2}\right)$$

$$= \pm\, a^3 \frac{3}{2\sqrt{2}}\cosh\frac{t}{2}.$$

This means that the torsion function is

$$\tau\,(t) = \pm\,\frac{3\sqrt{2}\cosh\dfrac{t}{2}}{a\left(3\cosh t + 1\right)} \tag{4.10}$$

(Figure 4. 2). Consequently, the Frenet equations are valid.

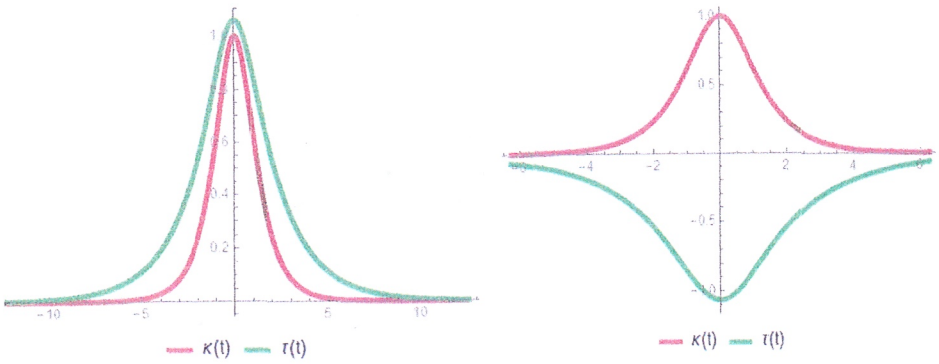



Fig. 4. 2: Curvature and torsion functions of Lorentzian Viviani curve II, $a = 1$.

**Corollary 4. 2** The Frenet instantaneous rotation vector of the frame $\{T(t), N(t), B(t)\}$ of the curve $\alpha_2(t)$ can be written in the form

$$f_2(t) = \pm 3\sqrt{2}\,\frac{\cosh\frac{t}{2}\sqrt{\cosh t}}{3\cosh t+1}\,T(t) - \frac{1}{2}\,\frac{\sqrt{3\cosh t+1}}{\cosh t}\,B(t). \qquad (4.11)$$

**Corollary 4. 3** The Frenet formulae of the spacelike curve $\alpha_2(t)$ can be written in the form

$$T'(t) = f_2(t) \times T(t),\ N'(t) = f_2(t) \times N(t),\ B'(t) = f_2(t) \times B(t)$$

by the vector $f_2(t)$ (Figure 4. 3).

**Proof.** It is sufficient to multiply the Frenet vectors by $f_2(t)$ vectorically in the Lorentz sense.

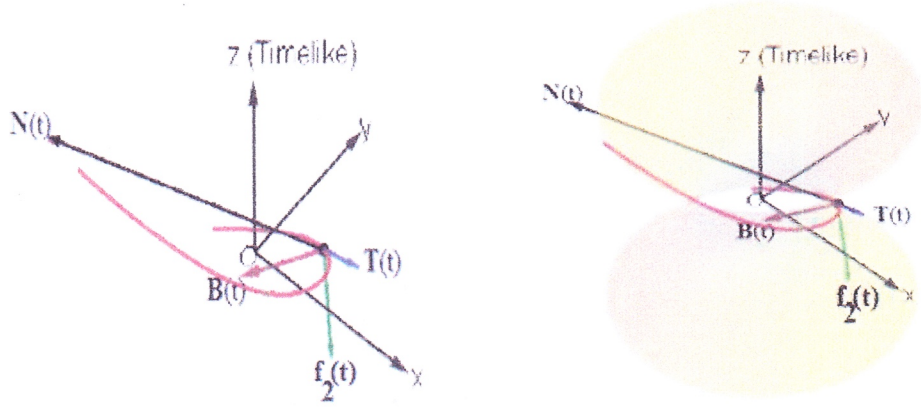



Fig.4.3: Frenet trihedron and the instantaneous rotation vector of the curve $\alpha_2(t)$

## 5 Lorentzian Viviani Curve III

In this chapter, we define the parametric equations and examine the geometry of the curve which is the intersection of the Lorentzian sphere of the radius $a$ centered at the origin with hyperbolic cylinder with radius $a$ centered at $(0, 0, a)$ at the Lorentz - Minkowski 3 - space.

A parameterization for this curve is given by

$$\alpha_3(t) = \left(a\sinh t,\ \pm a\sqrt{3+2\cosh t},\ a\left(1+\cosh t\right)\right),\ t \in R \tag{5.1}$$

where $a$ is positive. The curve defined by (5.1) is called *Lorentziyen Viviani eğrisi III.* The curve is on upper sheed of the hyperbolic cylinder (Figure 5.1).

For this curve, we have

$$v_3(t) = \left\| \alpha_3'(t) \right\| = a\sqrt{\frac{4\cosh t + \cosh 2t + 5}{2\left(3+2\cosh t\right)}} \tag{5.2}$$

from

$$\alpha_3'(t) = \left( a\cosh t,\ \pm a\frac{\sinh t}{\sqrt{3+2\cosh t}},\ a\sinh t \right). \tag{5.3}$$

Since $\cosh t \geq 1$ and $\cosh 2t \geq 1$; by (5.2), $\alpha_3(t)$ is a spacelike curve and it is not parametrized by the arc length.

If the second component of the curve $\alpha_3(t)$ is taken as +, the red colored curve arc is obtained and if it is taken as - , the green curve arc is obtained, as seen in figüre (5.1).

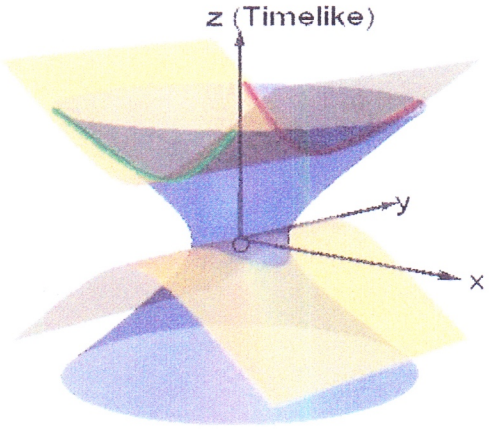



Fig. 5. 1: Lorentzian Viviani curve III, $a=1$.

**Theorem 5. 1** Let $\alpha_3 : I \subseteq R \to S_1^2(a)$ be regular Lorentzian Viviani curve III with arbitrary parameter t. In this case, the Frenet formulae of the Frenet frame $\{T, N, B\}$ are given by

$$\left.\begin{aligned}
T'(t) &= \frac{\sqrt{2\cosh^2\frac{t}{2}\left(3\cosh 3t+22\cosh 2t+85\cosh t+90\right)}}{\left(4\cosh t+\cosh 2t+5\right)\sqrt{3+2\cosh t}}N(t)\\
N'(t) &= \frac{\sqrt{2\cosh^2\frac{t}{2}\left(3\cosh 3t+22\cosh 2t+85\cosh t+90\right)}}{\left(4\cosh t+\cosh 2t+5\right)\sqrt{3+2\cosh t}}T(t)\mp\\
&\quad\frac{\frac{3\sqrt{2}}{2}\sqrt{4\cosh t+\cosh 2t+5}\left(5\sinh\frac{3t}{2}+\sinh\frac{5t}{2}\right)\operatorname{sech}\frac{t}{2}}{85\cosh t+22\cosh 2t+3\cosh 3t+90}B(t)\\
B'(t) &= \mp\frac{\frac{3\sqrt{2}}{2}\sqrt{4\cosh t+\cosh 2t+5}\left(5\sinh\frac{3t}{2}+\sinh\frac{5t}{2}\right)\operatorname{sech}\frac{t}{2}}{85\cosh t+22\cosh 2t+3\cosh 3t+90}N(t)
\end{aligned}\right\}. \quad (5.4)$$

**Proof.** By (5.2) and (5.3) the unit tangent vector field is

$$T(t)=\frac{1}{\sqrt{\dfrac{4\cosh t+\cosh 2t+5}{2(3+2\cosh t)}}}\left(\cosh t,\pm\frac{\sinh t}{\sqrt{3+2\cosh t}},\sinh t\right).$$

by a direct calculations, we have

$$\alpha_3''(t)=a\left(\sinh t,\pm\frac{\cosh t\sqrt{3+2\cosh t}-\dfrac{\sinh^2 t}{\sqrt{3+2\cosh t}}}{3+2\cosh t},\cosh t\right)$$

$$\alpha_3''(t)=a\left(\sinh t,\pm\frac{6\cosh t+\cosh 2t+3}{2(3+2\cosh t)^{3/2}},\cosh t\right),$$

$$\alpha_3'(t)\times\alpha_3''(t)=a^2\begin{vmatrix}-e_1 & -e_2 & e_3\\ \cosh t & \pm\dfrac{\sinh t}{\sqrt{3+2\cosh t}} & \sinh t\\ \sinh t & \pm\dfrac{6\cosh t+\cosh 2t+3}{2(3+2\cosh t)^{3/2}} & \cosh t\end{vmatrix}$$

$$=a^2\left(\pm\frac{6\cosh t\sinh t+\cosh 2t\sinh t+3\sinh t}{2(3+2\cosh t)^{3/2}}\mp\frac{\sinh t\cosh t}{\sqrt{3+2\cosh t}},1,\right.$$

$$\left.\pm\frac{6\cosh^2 t+\cosh 2t\cosh t+3\cosh t}{2(3+2\cosh t)^{3/2}}\mp\frac{\sinh^2 t}{\sqrt{3+2\cosh t}}\right)$$

$$\boldsymbol{\alpha}_3'(t)\times\boldsymbol{\alpha}_3''(t)=a^2\left(\mp\frac{\sinh^3 t}{(3+2\cosh t)^{3/2}},1,\mp\frac{\cosh 3t-9\cosh t-12}{4(3+2\cosh t)^{3/2}}\right), \tag{5.5}$$

$$\|\boldsymbol{\alpha}_3'(t)\times\boldsymbol{\alpha}_3''(t)\|=a^2\sqrt{\frac{\cosh^2\frac{t}{2}(3\cosh 3t+22\cosh 2t+85\cosh t+90)}{2(3+2\cosh t)^3}} \tag{5.6}$$

and

$$\langle\boldsymbol{\alpha}_3'(t)\times\boldsymbol{\alpha}_3''(t),\boldsymbol{\alpha}_3'(t)\times\boldsymbol{\alpha}_3''(t)\rangle=a^4\left(\frac{\sinh^6 t}{(3+2\cosh t)^3}+1-\frac{(\cosh 3t-9\cosh t-12)^2}{16(3+2\cosh t)^3}\right)$$

$$=\frac{a^4}{16(3+2\cosh t)^3}\big(16\sinh^6 t+16(3+2\cosh t)^3-\cosh^2 3t-81\cosh^2 t-144+$$

$$18\cosh 3t\cosh t+24\cosh 3t-108\cosh t\big)$$

$$=\frac{a^4\cosh^2\frac{t}{2}(3\cosh 3t+22\cosh 2t+85\cosh t+90)}{2(3+2\cosh t)^3}.$$

Thus, the curvature function of the curve $\alpha_3(t)$ is given by

$$\kappa(t)=\frac{a^2\sqrt{\frac{\cosh^2\frac{t}{2}(3\cosh 3t+22\cosh 2t+85\cosh t+90)}{2(3+2\cosh t)^3}}}{a^3\left(\frac{4\cosh t+\cosh 2t+5}{2(3+2\cosh t)}\right)^{3/2}}$$

or

$$\kappa(t)=\frac{2\cosh\frac{t}{2}\sqrt{3\cosh 3t+22\cosh 2t+85\cosh t+90}}{a\left(4\cosh t+\cosh 2t+5\right)^{3/2}}. \tag{5.7}$$

By (5.5) and (5.6), unit binormal vector field is found in this form

$$\boldsymbol{B}(t)=\frac{\sqrt{2}}{\cosh\frac{t}{2}\sqrt{3\cosh 3t+22\cosh 2t+85\cosh t+90}}\Big(\pm\sinh^3 t,\ -\left(3+2\cosh t\right)^{3/2},\ \pm\frac{1}{4}\left(\cosh 3t-9\cosh t-12\right)\Big).$$

The cross product $\boldsymbol{B}(t)\times\boldsymbol{T}(t)$ give principle normal vector field

$$\boldsymbol{N}(t)=\frac{2\sqrt{3+2\cosh t}}{\cosh\frac{t}{2}\sqrt{4\cosh t+\cosh 2t+5}\sqrt{3\cosh 3t+22\cosh 2t+85\cosh t+90}}\cdot$$

$$\begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \pm\sinh^3 t & -\left(3+2\cosh t\right)^{3/2} & \pm\frac{1}{4}\left(\cosh 3t-9\cosh t-12\right) \\ \cosh t & \pm\frac{\sinh t}{\sqrt{3+2\cosh t}} & \sinh t \end{vmatrix}$$

or

$$N(t)=\frac{1}{\cosh\frac{t}{2}\sqrt{4\cosh t+\cosh 2t+5}\sqrt{3\cosh 3t+22\cosh 2t+85\cosh t+90}}\cdot$$

$$\left(\frac{1}{4}\left(56\sinh t+38\sinh 2t+8\sinh 3t+\sinh 4t\right),\ \pm\sqrt{3+2\cosh t}\left(6\cosh t+\cosh 2t+3\right),\right.$$

$$\left.\frac{1}{4}\left(96\cosh t+44\cosh 2t+8\cosh 3t+\cosh 4t+51\right)\right).$$

Since $\langle N(t),\ N(t)\rangle=-1$, the vector field $N(t)$ is timelike. Third derivative of the curve is found in the form

$$\alpha_3'''(t)=a\Big(\cosh t,$$

$$\pm\frac{\left(6\sinh t+2\sinh 2t\right)2\left(3+2\cosh t\right)^{3/2}-\left(\cosh 2t+6\cosh t+3\right)6\sinh t\sqrt{3+2\cosh t}}{4\left(3+2\cosh t\right)^{3}},\ \sinh t\Big)$$

$$=a\left(\cosh t,\ \pm a\,\frac{6\sinh t+2\sinh 2t}{2\left(3+2\cosh t\right)^{3/2}}-\frac{3\sinh t\left(\cosh 2t+6\cosh t+3\right)}{2\left(3+2\cosh t\right)^{5/2}},\ \sinh t\right)$$

or if the necessary simplifications are made in the second component of this vector,

$$\alpha_3'''(t)=a\left(\cosh t,\ \pm\frac{\sinh(3t)+6\sinh(2t)+25\sinh t}{4\left(3+2\cosh t\right)^{5/2}},\ \sinh t\right).$$

The determinant of the first three derivatives and torsion function are given by

$$\det\left(\alpha_3'(t),\alpha_3''(t),\alpha_3'''(t)\right)=-\left\langle\alpha_3'(t)\times\alpha_3''(t),\alpha_3'''(t)\right\rangle$$

$$= -\left\langle a^2\left(\mp\frac{\sinh^3 t}{(3+2\cosh t)^{3/2}},\ 1,\ \mp\frac{\cosh 3t-9\cosh t-12}{4(3+2\cosh t)^{3/2}}\right),\right.$$
$$\left.\left(\cosh t,\ \pm\frac{\sinh 3t+6\sinh 2t+25\sinh t}{4(3+2\cosh t)^{5/2}},\ \sinh t\right)\right\rangle$$

$$= a^3\left(\pm\frac{\cosh t\sinh^3 t}{(3+2\cosh t)^{3/2}}\mp\frac{\sinh 3t+6\sinh 2t+25\sinh t}{4(3+2\cosh t)^{5/2}}\mp\right.$$
$$\left.\frac{\sinh t\cosh 3t-9\sinh t\cosh t-12\sinh t}{4(3+2\cosh t)^{3/2}}\right)$$

$$= \mp\ a^3\frac{1}{4(3+2\cosh t)^{5/2}}\left(-4(3+2\cosh t)\cosh t\sinh^3 t+\sinh 3t+\sinh 2t\right.$$
$$\left.+25\sinh t+(3+2\cosh t)(\sinh t\cosh 3t-9\sinh t\cosh t-12\sinh t)\right)$$

$$= \mp\ a^3\frac{3\sinh t(3+6\cosh t+\cosh 2t)}{2(3+2\cosh t)^{5/2}}$$

and

$$\tau(t) = \frac{\mp\ a^3\dfrac{3\sinh t(3+6\cosh t+\cosh 2t)}{2(3+2\cosh t)^{5/2}}}{a^4\dfrac{\cosh^2\dfrac{t}{2}(3\cosh 3t+22\cosh 2t+85\cosh t+90)}{2(3+2\cosh t)^3}}$$

$$= \mp\frac{6\sqrt{3+2\cosh t}\,(3+6\cosh t+\cosh 2t)\tanh\dfrac{t}{2}}{a(3\cosh 3t+22\cosh 2t+85\cosh t+90)}$$

$$\tau(t)=\mp\frac{3\operatorname{sech}\frac{t}{2}\sqrt{3+2\cosh t}\left(5\sinh\frac{3t}{2}+\sinh\frac{5t}{2}\right)}{a\left(85\cosh t+22\cosh 2t+3\cosh 3t+90\right)} \tag{5.8}$$

respectively. At the minumum point $\alpha_3(0)=\left(0,\pm a\sqrt{5},2a\right)$ for $t=0$, torsion is zero (Figure 5.2).

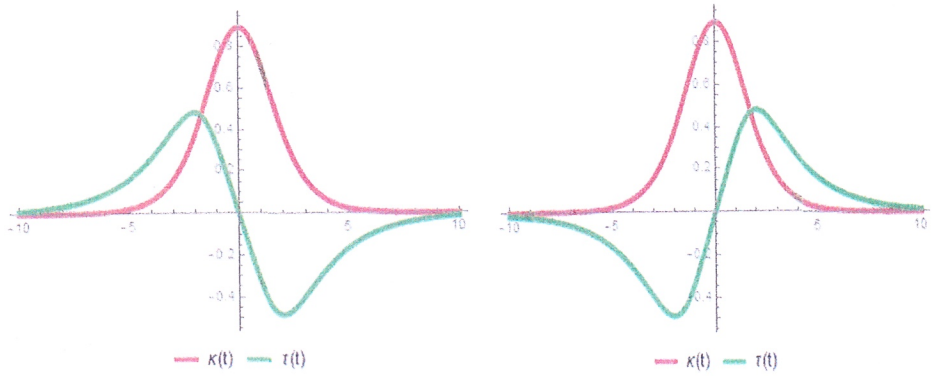



Fig. 5. 2: Curvature and torsion functions of the Lorentzian Viviani curve III, $a=1$.

**Corollary 5. 2** The Frenet instantaneous rotation vector of the frame $\{T(t),N(t),B(t)\}$ of the curve $\alpha_3(t)$ can be written in the form

$$f_3(t)=\pm\frac{\frac{3\sqrt{2}}{2}\operatorname{sech}\frac{t}{2}\sqrt{4\cosh t+\cosh 2t+5}\left(5\sinh\frac{3t}{2}+\sinh\frac{5t}{2}\right)}{85\cosh t+22\cosh 2t+3\cosh 3t+90}T(t)+$$

$$\frac{\sqrt{2\cosh^2\frac{t}{2}\left(3\cosh 3t+22\cosh 2t+85\cosh t+90\right)}}{\sqrt{3+2\cosh t}\left(4\cosh t+\cosh 2t+5\right)}B(t) \tag{5.9}$$

(Figure 5. 3).

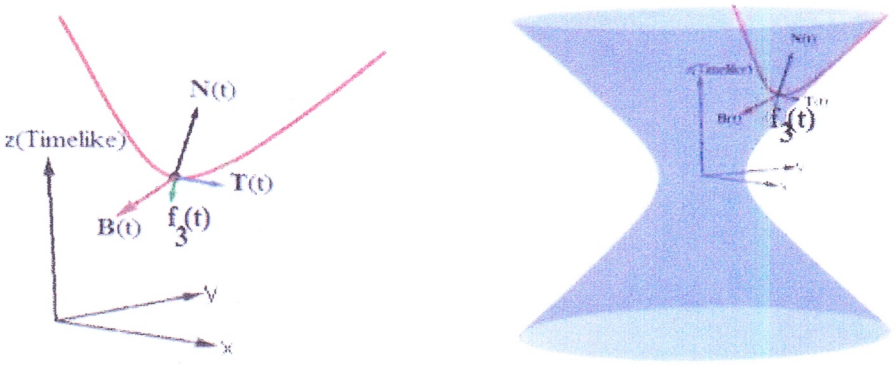



Fig. 5. 3: Frenet trihedron and instantaneous rotation vector of the curve $\alpha_3(t)$

**Corollary 5. 3** The Frenet formulas of the timelike arc of the curve $\alpha_3(t)$ can be written in the form

$$\boldsymbol{T'(t) = f_3(t) \times T(t)},\ \boldsymbol{N'(t) = f_3(t) \times N(t)},\ \boldsymbol{B'(t) = f_3(t) \times B(t)}$$

by the vector $f_3(t)$.

**Proof.** It is sufficient to multiply the Frenet vectors by $f_3(t)$ vectorically in the Lorentz sense.

## 6 Lorentziyen Viviani Curve IV

In this chapter, we define the parametric equations and examine the geometry of the curve which is the intersection of the Lorentzian sphere of the radius $a$ centered at the origin with hyperbolic cylinder with radius $a$ centered at $(0, 0, a)$ at the Lorentz - Minkowski 3 - space.

The parametric equations of the lower sheed of the hyperbolic cylinder are given by

$$x = a\sinh t\ ,\ z = a - a\cosh t\ . \tag{6.1}$$

If these equations are written in equation of the sphere, it seen that

$$y = \pm\, a\sqrt{3-2\cosh t}\,. \tag{6.2}$$

By (6.1) and (6.2), the parametrization for this curve is given by

$$\alpha_4(t) = \left(a\sinh t,\ \pm\, a\sqrt{3-2\cosh t},\ a\left(1-\cosh t\right)\right), \tag{6.3}$$

where $t \in \left[-\cosh^{-1}\frac{3}{2}, \cosh^{-1}\frac{3}{2}\right]$. The curve defined by (6.3) is called *Lorentzian Viviani curve IV* . The curve is on lower sheed of the hyperbolic cylinder (Figure 6.2) , by (6.4), it is not parametrized by arc lenght and is spacelike curve as seen in figüre 6.1.

For this curve, we have

$$v_4(t) = \left\| \alpha_4'(t) \right\| = a\sqrt{\frac{-4\cosh t + \cosh 2t + 5}{2\left(3-2\cosh t\right)}} \tag{6.4}$$

from

$$\alpha_4'(t) = \left( a\cosh t,\ \mp\, a\,\frac{\sinh t}{\sqrt{3-2\cosh t}},\ -a\sinh t \right). \tag{6.5}$$

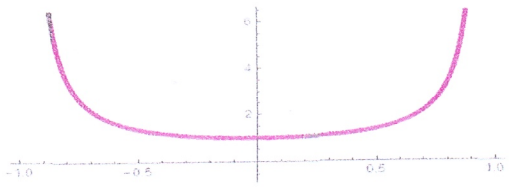



Fig. 6. 1: The graphic of the function $\left\langle \alpha_4'(t),\ \alpha_4'(t) \right\rangle$; $t \in \left[-\cosh^{-1}\frac{3}{2},\ \cosh^{-1}\frac{3}{2}\right]$ , $a = 1$.

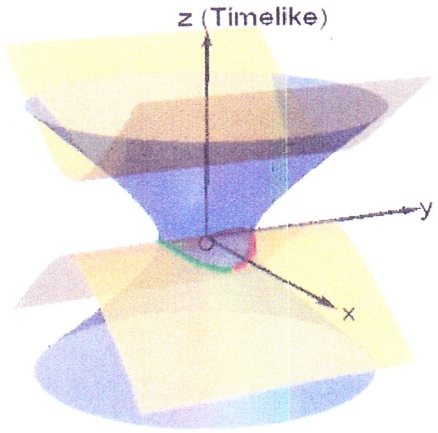
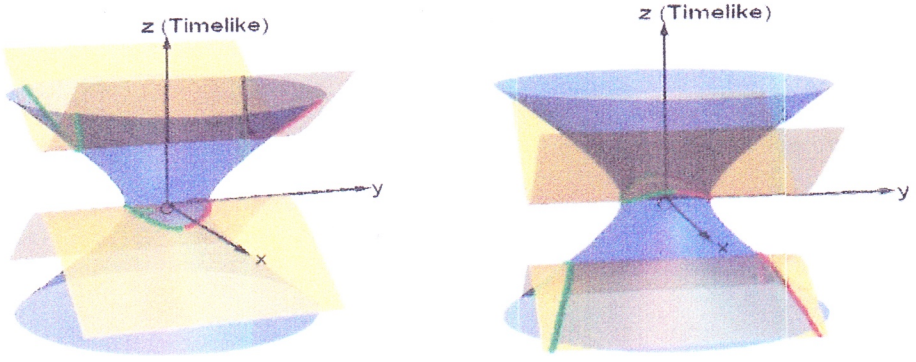



Fig. 6. 2: Lorentzian Viviani curve IV , $a=1$.

If the second component of the curve $\alpha_4(t)$ given by (6.3) is taken as +, the red curve arc on the lower sheed of the cylinder is found, and if it is taken as -, the green curve arc is found, as seen in figure 6.2. If the Cartesian equation of the hyperbolic cylinder is taken as $-x^2+(z+a)^2=a^2$, the symmetry of the curves with respect to the $XY-$ plane is obtained, as seen in figure 6.3 (b).




(a) (b)

Fig. 6. 3: Lorentzian Viviani curve III and IV

**Theorem 6. 1** Let $\alpha_4 : I \subseteq R \to S_1^2(a)$ be regular Lorentzian Viviani curve IV given by (6.3). In this case, the Frenet equations of the Frenet frame $\{T, N, B\}$ of the curve $\alpha_4(t)$ are given by

$$\left.\begin{aligned}
T'(t) &= \frac{\sqrt{2\sinh^2\frac{t}{2}\left(-3\cosh 3t + 22\cosh 2t - 85\cosh t + 90\right)}}{\left(-4\cosh t + \cosh 2t + 5\right)\sqrt{3-2\cosh t}} N(t) \\
N'(t) &= -\frac{\sqrt{2\sinh^2\frac{t}{2}\left(-3\cosh 3t + 22\cosh 2t - 85\cosh t + 90\right)}}{\left(-4\cosh t + \cosh 2t + 5\right)\sqrt{3-2\cosh t}} T(t) \pm \\
&\quad \frac{\frac{3\sqrt{2}}{2}\sqrt{-4\cosh t + \cosh 2t + 5}\left(-5\cosh\frac{3t}{2} + \cosh\frac{5t}{2}\right)\operatorname{csch}\frac{t}{2}}{-3\cosh 3t + 22\cosh 2t - 85\cosh t + 90} B(t) \\
B'(t) &= \pm \frac{\frac{3\sqrt{2}}{2}\sqrt{-4\cosh t + \cosh 2t + 5}\left(-5\cosh\frac{3t}{2} + \cosh\frac{5t}{2}\right)\operatorname{csch}\frac{t}{2}}{-3\cosh 3t + 22\cosh 2t - 85\cosh t + 90} N(t).
\end{aligned}\right\} \quad (6.6)$$

**Proof.** By (6.4) and (6.5) the unit tangent vector field is

$$T(t) = \frac{1}{\sqrt{\frac{-4\cosh t + \cosh 2t + 5}{2(3-2\cosh t)}}}\left(\cosh t,\ \mp\frac{\sinh t}{\sqrt{3-2\cosh t}}, -\sinh t\right).$$

By direct calculations, we have

$$\alpha_4''(t) = a\left(\sinh t,\ \mp\frac{\cosh t\sqrt{3-2\cosh t} + \frac{\sinh^2 t}{\sqrt{3-2\cosh t}}}{3-2\cosh t}, -\cosh t\right)$$

$$\boldsymbol{\alpha}_4''(t) = a\left(\sinh t, \mp \frac{-\cosh 2t + 6\cosh t - 3}{2(3-2\cosh t)^{3/2}}, -\cosh t\right),$$

$$\boldsymbol{\alpha}_4'(t)\times\boldsymbol{\alpha}_4''(t) = a^2\begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \cosh t & \mp\dfrac{\sinh t}{\sqrt{3-2\cosh t}} & -\sinh t \\ \sinh t & \mp\dfrac{6\cosh t - \cosh 2t - 3}{2(3-2\cosh t)^{3/2}} & -\cosh t \end{vmatrix}$$

$$= a^2\left(\mp\frac{-6\cosh t\sinh t + \cosh 2t\sinh t + 3\sinh t}{2(3-2\cosh t)^{3/2}} \mp \frac{\sinh t\cosh t}{\sqrt{3-2\cosh t}}, -1,\right.$$
$$\left. \mp\frac{6\cosh^2 t - \cosh 2t\cosh t - 3\cosh t}{2(3-2\cosh t)^{3/2}} \pm \frac{\sinh^2 t}{\sqrt{3-2\cosh t}}\right)$$

$$= a^2\left(\pm\frac{\sinh^3 t}{(3-2\cosh t)^{3/2}}, -1, \mp\frac{\cosh 3t - 9\cosh t + 12}{4(3-2\cosh t)^{3/2}}\right),$$

$$\|\boldsymbol{\alpha}_4'(t)\times\boldsymbol{\alpha}_4''(t)\| = a^2\sqrt{\left|\frac{\sinh^2\frac{t}{2}(3\cosh 3t - 22\cosh 2t + 85\cosh t - 90)}{2(3-2\cosh t)^3}\right|}$$

$$= a^2\sqrt{\frac{\sinh^2\frac{t}{2}(-3\cosh 3t + 22\cosh 2t - 85\cosh t + 90)}{2(3-2\cosh t)^3}},$$

and

$$\langle \alpha_4'(t)\times\alpha_4''(t),\ \alpha_4'(t)\times\alpha_4''(t)\rangle = \frac{a^4\sinh^2\frac{t}{2}\left(3\cosh 3t-22\cosh 2t+85\cosh t-90\right)}{2\left(3-2\cosh t\right)^3}$$

(Figure 6. 4).

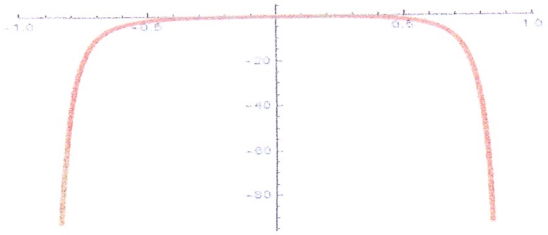


Fig. 6.4: The graphic of function $\langle \alpha_4'(t)\times\alpha_4''(t),\ \alpha_4'(t)\times\alpha_4''(t)\rangle$, $t\in\left[-\cosh^{-1}\frac{3}{2},\cosh^{-1}\frac{3}{2}\right]$, $a=1$.

Accordingly, null frame is defined at $t=0\ (\alpha_4(0)=(0,\pm1,0))$ and for $t\neq 0$, the unit principle normal and unit timelike binormal vector fields are found in the forms

$$\mathrm{N(t)} = \frac{2\sqrt{3-2\cosh t}}{\sqrt{-4\cosh t+\cosh 2t+5}\sqrt{\sinh^2\frac{t}{2}\left(-3\cosh 3t+22\cosh 2t-85\cosh t+90\right)}}\cdot$$

$$\begin{vmatrix} -e_1 & -e_2 & e_3 \\ \cosh t & \mp\dfrac{\sinh t}{\sqrt{3-2\cosh t}} & -\sinh t \\ \pm\sinh^3 t & -\left(3-2\cosh t\right)^{3/2} & \mp\dfrac{1}{4}\left(\cosh 3t-9\cosh t+12\right) \end{vmatrix}.$$

$$N(t)=\frac{1}{\sqrt{-4\cosh t+\cosh 2t+5}\sqrt{\sinh^2\frac{t}{2}\left(-3\cosh 3t+22\cosh 2t-85\cosh t+90\right)}}$$

$$\left(\frac{-56\sinh t+38\sinh 2t-8\sinh 3t+\sinh 4t}{4},\ \mp\sqrt{3-2\cosh t}\left(3-6\cosh t+\cosh 2t\right),\right.$$

$$\left.-\frac{1}{4}\left(-96\cosh t+44\cosh 2t-8\cosh 3t+\cosh 4t+51\right)\right),$$

and

$$\boldsymbol{B}(t)=\frac{\sqrt{2}\left(3-2\cosh t\right)^{3/2}}{\sqrt{\sinh^2\frac{t}{2}\left(-3\cosh 3t+22\cosh 2t-85\cosh t+90\right)}}\left(\pm\frac{\sinh^3 t}{\left(3-2\cosh t\right)^{3/2}},\right.$$

$$\left.-1,\ \mp\frac{\cosh 3t-9\cosh t+12}{4\left(3-2\cosh t\right)^{3/2}}\right)$$

$$\boldsymbol{B}(t)=\frac{\sqrt{2}}{\sqrt{\sinh^2\frac{t}{2}\left(-3\cosh 3t+22\cosh 2t-85\cosh t+90\right)}}\left(\pm\sinh^3 t,\right.$$

$$\left.-\left(3-2\cosh t\right)^{3/2},\ \mp\frac{1}{4}\left(\cosh 3t-9\cosh t+12\right)\right),$$

respectively.

Therefore, the curvature function is

$$\kappa(t)=\frac{a^2\sqrt{\sinh^2\frac{t}{2}\left(-3\cosh 3t+22\cosh 2t-85\cosh t+90\right)}}{a^3\left(\frac{-4\cosh t+\cosh 2t+5}{2\left(3-2\cosh t\right)}\right)^{3/2}}$$

$$\kappa(t)=\frac{2\sqrt{\sinh^2\frac{t}{2}\left(-3\cosh 3t+22\cosh 2t-85\cosh t+90\right)}}{a\left(-4\cosh t+\cosh 2t+5\right)^{3/2}}. \qquad (6.7)$$

Let us the third derivative of the curve:

$$\alpha_4'''(t)=a\Big(\cosh t,$$

$$\mp\frac{\left(6\sinh t-2\sinh 2t\right)2\left(3-2\cosh t\right)^{3/2}-\left(-\cosh 2t+6\cosh t-3\right)\left(-6\sinh t\right)\sqrt{3-2\cosh t}}{4\left(3-2\cosh t\right)^3},$$

$$-\sinh t\Big)$$

$$=a\left(\cosh t,\mp\frac{6\sinh t-2\sinh 2t}{2\left(3-2\cosh t\right)^{3/2}}+\frac{3\sinh t\left(-\cosh 2t+6\cosh t-3\right)}{2\left(3-2\cosh t\right)^{5/2}},-\sinh t\right)$$

$$=a\left(\cosh t,\pm\frac{\sinh 3t-6\sinh 2t+25\sinh t}{4\left(3-2\cosh t\right)^{5/2}},-\sinh t\right).$$

The determinant of the first three derivatives and torsion function are given by

$$\det\left(\boldsymbol{\alpha}_4'(t),\boldsymbol{\alpha}_4''(t),\boldsymbol{\alpha}_4'''(t)\right)=-\left\langle\boldsymbol{\alpha}_4'(t)\times\boldsymbol{\alpha}_4''(t),\boldsymbol{\alpha}_4'''(t)\right\rangle$$

$$\det\left(\boldsymbol{\alpha}_4'(t),\boldsymbol{\alpha}_4''(t),\boldsymbol{\alpha}_4'''(t)\right) = -\left\langle a^2\left(\pm\frac{\sinh^3 t}{(3-2\cosh t)^{3/2}},-1,\mp\frac{\cosh 3t-9\cosh t+12}{4(3-2\cosh t)^{3/2}}\right),\right.$$
$$\left. a\left(\cosh t,\pm\frac{\sinh 3t-6\sinh 2t+25\sinh t}{4(3-2\cosh t)^{5/2}},-\sinh t\right)\right\rangle$$

$$\det(\boldsymbol{\alpha}_4'(t),\boldsymbol{\alpha}_4''(t),\boldsymbol{\alpha}_4'''(t)) = \mp a^3\frac{3\sinh t(3-6\cosh t+\cosh 2t)}{2(3-2\cosh t)^{5/2}}.$$

Thus, by (2.9), the torsion function is given by

$$\tau(t) = -\frac{\mp a^3\dfrac{3\sinh t(3-6\cosh t+\cosh 2t)}{2(3-2\cosh t)^{5/2}}}{a^4\dfrac{\sinh^2\dfrac{t}{2}(-3\cosh 3t+22\cosh 2t-85\cosh t+90)}{2(3-2\cosh t)^3}}$$

or

$$\tau(t) = \pm\frac{3\operatorname{csch}\dfrac{t}{2}\sqrt{3-2\cosh t}\left(-5\cosh\dfrac{3t}{2}+\cosh\dfrac{5t}{2}\right)}{a(-3\cosh 3t+22\cosh 2t-85\cosh t+90)}. \qquad (6.8)$$

For $t=\pm\cosh^{-1}\dfrac{3}{2}$, the torsion at the points $\alpha_4\left(\pm\cosh^{-1}\dfrac{3}{2}\right)=a\left(\pm\dfrac{\sqrt{5}}{2},0,-\dfrac{1}{2}\right)$ is zero (Figure 6. 5) and is undefined for $t=0$. The invariants (6.7) and (6.8) satisfies the Frenet derivative formulas (6.6).

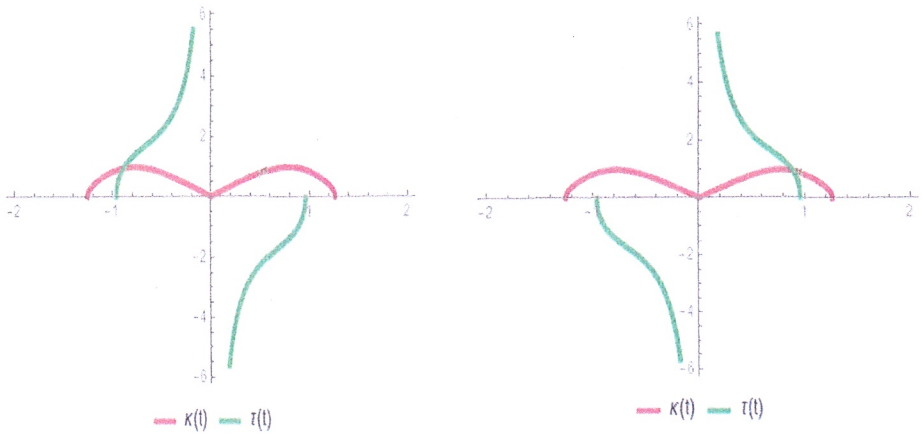



Fig. 6. 5:Curvature and torsion functions of the Lorentziyen Viviani Curve IV, $t \in \left[-\cosh^{-1}\frac{3}{2}, \cosh^{-1}\frac{3}{2}\right]$, $a=1$.

**Corollary 6. 2** By (6.7) and (6.8), the Frenet instantaneous rotation vector of the Frenet frame $\{T, N, B\}$ of the Lorentzian Viviani curve IV can be written in the form

$$f_4(t) = \pm \frac{\frac{3\sqrt{2}}{2}\operatorname{csch}\frac{t}{2}\sqrt{-4\cosh t+\cosh 2t+5}\left(-5\cosh\frac{3t}{2}+\cosh\frac{5t}{2}\right)}{-3\cosh 3t+22\cosh 2t-85\cosh t+90} T(t) - \frac{\sqrt{2\sinh^2\frac{t}{2}\left(-3\cosh 3t+22\cosh 2t-85\cosh t+90\right)}}{\sqrt{3-2\cosh t}\left(-4\cosh t+\cosh 2t+5\right)} B(t) \tag{6.9}$$

(Figure 6. 6).

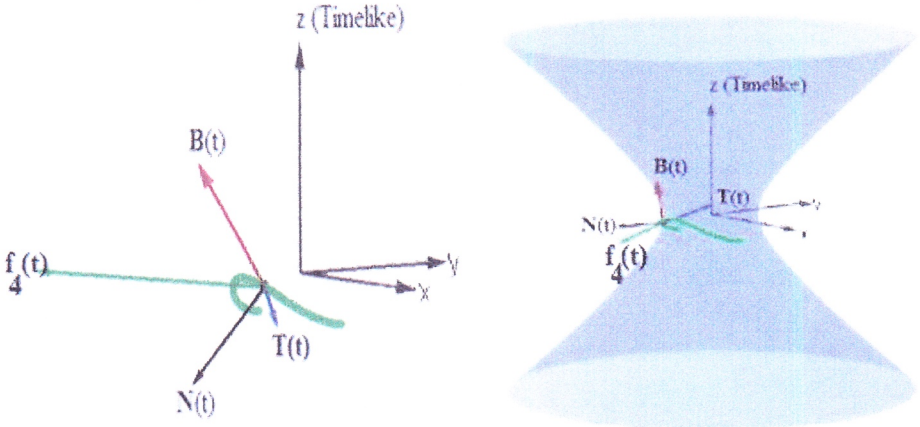



Fig. 6. 6: The Frenet frame and the instantaneous rotation vector of the Lorentzian Viviani curve IV.

**Corollary 6. 3** The Frenet formulas of the spacelike curve $\alpha_4(t)$ are given by

$$T'(t) = f_4(t) \times T(t),\ N'(t) = f_4(t) \times N(t),\ B'(t) = f_4(t) \times B(t)$$

by the vector $f_4(t)$. (Figure 6. 6)

**Proof.** It is sufficient to multiply the Frenet vectors by $f_4(t)$ vectorically in the Lorentz sense.

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